EJERCICIO (21:14)

Demostrar que:

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = \gamma^{\mu}p_{\mu} + m$$

Donde:

$$u_{1\,(p)} = \sqrt{E_p + m}\begin{pmatrix} \cos{}^{\theta}/_{2} \\ e^{i\phi}\sin{}^{\theta}/_{2} \\ \frac{|\vec{p}|}{E_p+m}\cos{}^{\theta}/_{2} \\ \frac{|\vec{p}|}{E_p+m}e^{i\phi}\sin{}^{\theta}/_{2} \end{pmatrix} \qquad u_{2\,(p)} = \sqrt{E_p + m}\begin{pmatrix} -\sin{}^{\theta}/_{2} \\ e^{i\phi}\cos{}^{\theta}/_{2} \\ \frac{|\vec{p}|}{E_p+m}\sin{}^{\theta}/_{2} \\ -\frac{|\vec{p}|}{E_p+m}e^{i\phi}\cos{}^{\theta}/_{2} \end{pmatrix}$$

En el formulario https://crul.github.io/CursoTeoriaCuanticaDeCamposJavierGarcia/ (Crul, Roger, Sware), se presentan los estados de helicidad, ver fórmula 42.2:

$$\chi_+ = \begin{pmatrix} \cos{}^{\theta}/_{2} \\ e^{i\phi}\sin{}^{\theta}/_{2} \end{pmatrix}$$

$$\chi_- = \begin{pmatrix} -e^{i\phi}\sin{}^{\theta}/_{2} \\ \cos{}^{\theta}/_{2} \end{pmatrix}$$

Expresando los espinores en función de estos estados:

$$u_{1\,(p)} = \sqrt{E_p + m}\begin{pmatrix} \chi_+ \\ \frac{|\vec{p}|}{E_p + m}\chi_+ \end{pmatrix}$$

Como: $e^{i\phi}\chi_- = e^{i\phi}\begin{pmatrix} -e^{i\phi}\sin{}^{\theta}/_{2} \\ \cos{}^{\theta}/_{2} \end{pmatrix} = \begin{pmatrix} -\sin{}^{\theta}/_{2} \\ e^{i\phi}\cos{}^{\theta}/_{2} \end{pmatrix}$

$$u_{2\,(p)} = \sqrt{E_p + m}\begin{pmatrix} e^{i\phi}\chi_- \\ -\frac{|\vec{p}|}{E_p + m}e^{i\phi}\chi_- \end{pmatrix} = \sqrt{E_p + m}\, e^{i\phi}\begin{pmatrix} \chi_- \\ -\frac{|\vec{p}|}{E_p + m}\chi_- \end{pmatrix}$$

Sabemos que: $\overline{u_{r\,(p)}} = {u_{r\,(p)}}^{\dagger}\,\gamma^0$

Debemos calcular primero los transpuestos conjugados de los espinores:

$${u_{1\,(p)}}^{\dagger} = \sqrt{E_p + m}\,\begin{pmatrix} {\chi_+}^{\dagger} & \frac{|\vec{p}|}{E_p + m}{\chi_+}^{\dagger} \end{pmatrix}$$

$${u_{2\,(p)}}^{\dagger} = \sqrt{E_p + m}\, e^{-i\phi}\,\begin{pmatrix} {\chi_-}^{\dagger} & -\frac{|\vec{p}|}{E_p + m}{\chi_-}^{\dagger} \end{pmatrix}$$

Calculamos los dos términos de $u_{r\,(p)}{u_{r\,(p)}}^\dagger$

$$u_{1\,(p)}{u_{1\,(p)}}^\dagger = \sqrt{E_p+m}\begin{pmatrix}\chi_+\\ \frac{|\vec{p}|}{E_p+m}\chi_+\end{pmatrix}\sqrt{E_p+m}\begin{pmatrix}{\chi_+}^\dagger & \frac{|\vec{p}|}{E_p+m}{\chi_+}^\dagger\end{pmatrix}$$

$$u_{1\,(p)}{u_{1\,(p)}}^\dagger = (E_p+m)\begin{pmatrix}\chi_+\\ \frac{|\vec{p}|}{E_p+m}\chi_+\end{pmatrix}\begin{pmatrix}{\chi_+}^\dagger & \frac{|\vec{p}|}{E_p+m}{\chi_+}^\dagger\end{pmatrix}$$

$$u_{1\,(p)}{u_{1\,(p)}}^\dagger = (E_p+m)\begin{pmatrix}\chi_+{\chi_+}^\dagger & \chi_+\frac{|\vec{p}|}{E_p+m}{\chi_+}^\dagger\\ \frac{|\vec{p}|}{E_p+m}\chi_+{\chi_+}^\dagger & \frac{|\vec{p}|}{E_p+m}\chi_+\frac{|\vec{p}|}{E_p+m}{\chi_+}^\dagger\end{pmatrix}$$

$$u_{1\,(p)}{u_{1\,(p)}}^\dagger = (E_p+m)\begin{pmatrix}\chi_+{\chi_+}^\dagger & \frac{|\vec{p}|}{E_p+m}\chi_+{\chi_+}^\dagger\\ \frac{|\vec{p}|}{E_p+m}\chi_+{\chi_+}^\dagger & \left(\frac{|\vec{p}|}{E_p+m}\right)^2\chi_+{\chi_+}^\dagger\end{pmatrix}$$

$$u_{2\,(p)}{u_{2\,(p)}}^\dagger = \sqrt{E_p+m}\,e^{i\phi}\begin{pmatrix}\chi_-\\ -\frac{|\vec{p}|}{E_p+m}\chi_-\end{pmatrix}\sqrt{E_p+m}\,e^{-i\phi}\begin{pmatrix}{\chi_-}^\dagger & -\frac{|\vec{p}|}{E_p+m}{\chi_-}^\dagger\end{pmatrix}$$

$$u_{2\,(p)}{u_{2\,(p)}}^\dagger = (E_p+m)\begin{pmatrix}\chi_-\\ -\frac{|\vec{p}|}{E_p+m}\chi_-\end{pmatrix}\begin{pmatrix}{\chi_-}^\dagger & -\frac{|\vec{p}|}{E_p+m}{\chi_-}^\dagger\end{pmatrix}$$

$$u_{2\,(p)}{u_{2\,(p)}}^\dagger = (E_p+m)\begin{pmatrix}\chi_-{\chi_-}^\dagger & -\chi_-\frac{|\vec{p}|}{E_p+m}{\chi_-}^\dagger\\ -\frac{|\vec{p}|}{E_p+m}\chi_-{\chi_-}^\dagger & \frac{|\vec{p}|}{E_p+m}\chi_-\frac{|\vec{p}|}{E_p+m}{\chi_-}^\dagger\end{pmatrix}$$

$$u_{2\,(p)}{u_{2\,(p)}}^\dagger = (E_p+m)\begin{pmatrix}\chi_-{\chi_-}^\dagger & -\frac{|\vec{p}|}{E_p+m}\chi_-{\chi_-}^\dagger\\ -\frac{|\vec{p}|}{E_p+m}\chi_-{\chi_-}^\dagger & \left(\frac{|\vec{p}|}{E_p+m}\right)^2\chi_-{\chi_-}^\dagger\end{pmatrix}$$

Sumando ambos términos:

$$u_{1\,(p)}{u_{1\,(p)}}^\dagger + u_{2\,(p)}{u_{2\,(p)}}^\dagger =$$

$$= (E_p+m)\begin{pmatrix}\chi_+{\chi_+}^\dagger + \chi_-{\chi_-}^\dagger & \frac{|\vec{p}|}{E_p+m}\chi_+{\chi_+}^\dagger - \frac{|\vec{p}|}{E_p+m}\chi_-{\chi_-}^\dagger\\ \frac{|\vec{p}|}{E_p+m}\chi_+{\chi_+}^\dagger - \frac{|\vec{p}|}{E_p+m}\chi_-{\chi_-}^\dagger & \left(\frac{|\vec{p}|}{E_p+m}\right)^2\chi_+{\chi_+}^\dagger + \left(\frac{|\vec{p}|}{E_p+m}\right)^2\chi_-{\chi_-}^\dagger\end{pmatrix}$$

$$u_{1\,(p)}{u_{1\,(p)}}^{\dagger}+u_{2\,(p)}{u_{2\,(p)}}^{\dagger}=\left(E_p+m\right)\begin{pmatrix} \chi_+{\chi_+}^{\dagger}+\chi_-{\chi_-}^{\dagger} & \frac{|\vec{p}|}{E_p+m}\left(\chi_+{\chi_+}^{\dagger}-\chi_-{\chi_-}^{\dagger}\right) \\ \frac{|\vec{p}|}{E_p+m}\left(\chi_+{\chi_+}^{\dagger}-\chi_-{\chi_-}^{\dagger}\right) & \left(\frac{|\vec{p}|}{E_p+m}\right)^2\left(\chi_+{\chi_+}^{\dagger}+\chi_-{\chi_-}^{\dagger}\right) \end{pmatrix}$$

La fórmula 43.3 del formulario de Crul et al. previamente citado muestra la resolución de la identidad:

$\chi_+{\chi_+}^{\dagger}+\chi_-{\chi_-}^{\dagger}=\mathbb{I}$

Donde $\mathbb{I}$ es una matriz identidad de 2x2.

En el video del capítulo 43, minuto 33:40, Javier ha demostrado que:

$\chi_+{\chi_+}^{\dagger}-\chi_-{\chi_-}^{\dagger}=\vec{\sigma}\cdot\hat{n}$

Donde σ_i son las matrices de Pauli (fórmula 41.4 del formulario de Crul)

Resulta, entonces, que:

$$u_{1\,(p)}{u_{1\,(p)}}^{\dagger}+u_{2\,(p)}{u_{2\,(p)}}^{\dagger}=\left(E_p+m\right)\begin{pmatrix} \mathbb{I} & \frac{|\vec{p}|}{E_p+m}(\vec{\sigma}\cdot\hat{n}) \\ \frac{|\vec{p}|}{E_p+m}(\vec{\sigma}\cdot\hat{n}) & \left(\frac{|\vec{p}|}{E_p+m}\right)^2\mathbb{I} \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}}=u_{1\,(p)}{u_{1\,(p)}}^{\dagger}\gamma^0+u_{2\,(p)}{u_{2\,(p)}}^{\dagger}\gamma^0=\left(u_{1\,(p)}{u_{1\,(p)}}^{\dagger}+u_{2\,(p)}{u_{2\,(p)}}^{\dagger}\right)\gamma^0$$

$$\gamma^0=\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}}=\left(E_p+m\right)\begin{pmatrix} \mathbb{I} & \frac{|\vec{p}|}{E_p+m}(\vec{\sigma}\cdot\hat{n}) \\ \frac{|\vec{p}|}{E_p+m}(\vec{\sigma}\cdot\hat{n}) & \left(\frac{|\vec{p}|}{E_p+m}\right)^2\mathbb{I} \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}}=\left(E_p+m\right)\begin{pmatrix} \mathbb{I} & -\frac{|\vec{p}|}{E_p+m}(\vec{\sigma}\cdot\hat{n}) \\ \frac{|\vec{p}|}{E_p+m}(\vec{\sigma}\cdot\hat{n}) & -\left(\frac{|\vec{p}|}{E_p+m}\right)^2\mathbb{I} \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}}=\begin{pmatrix} \left(E_p+m\right)\mathbb{I} & -|\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) \\ |\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) & -\frac{|\vec{p}|^2}{E_p+m}\mathbb{I} \end{pmatrix}$$

Recordando que: ${E_p}^2 = |\vec{p}|^2 + m^2$

Resulta que: $|\vec{p}|^2 = {E_p}^2 - m^2 = (E_p - m)(E_p + m)$, en consecuencia:

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = \begin{pmatrix} (E_p + m)\,\mathbb{I} & -|\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) \\ |\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) & -\dfrac{(E_p - m)(E_p + m)}{E_p + m}\,\mathbb{I} \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = \begin{pmatrix} (E_p + m)\,\mathbb{I} & -|\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) \\ |\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) & -(E_p - m)\,\mathbb{I} \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = \begin{pmatrix} m\,\mathbb{I} & 0 \\ 0 & m\,\mathbb{I} \end{pmatrix} + \begin{pmatrix} E_p\,\mathbb{I} & 0 \\ 0 & -E_p\,\mathbb{I} \end{pmatrix} + \begin{pmatrix} 0 & -|\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) \\ |\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) & 0 \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = m\begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} + E_p\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} + \begin{pmatrix} 0 & -|\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) \\ |\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) & 0 \end{pmatrix}$$

Pero como:

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

Entonces:

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = m\,\mathbb{I} + E_p\,\gamma^0 + \begin{pmatrix} 0 & -|\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) \\ |\vec{p}|\,(\vec{\sigma}\cdot\hat{n}) & 0 \end{pmatrix}$$

Considerando que: $\vec{\sigma}\cdot\hat{n} = \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = m\,\mathbb{I} + E_p\,\gamma^0 + \begin{pmatrix} 0 & -|\vec{p}|\,\sigma_1 n_1 \\ |\vec{p}|\,\sigma_1 n_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -|\vec{p}|\,\sigma_2 n_2 \\ |\vec{p}|\sigma_2 n_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -|\vec{p}|\,\sigma_3 n_3 \\ |\vec{p}|\,\sigma_3 n_3 & 0 \end{pmatrix}$$

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = m\,\mathbb{I} + E_p\,\gamma^0 + |\vec{p}|n_1\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} + |\vec{p}|n_2\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} + |\vec{p}|n_3\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

Como: $|\vec{p}|n_i = p_i$ y $E_p = p_0$

Y las matrices gamma en la representación de Dirac son (fórmula 41.5, formulario Crul et al.):

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

Resulta que:

$$\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = m\,\mathbb{I} + p_0\,\gamma^0 - p_1\,\gamma^1 - p_2\,\gamma^2 - p_3\,\gamma^3$$

$$\boxed{\boldsymbol{\sum_{r=1}^{2} u_{r\,(p)}\overline{u_{r\,(p)}} = m + p_\mu\,\gamma^\mu}}$$

QED (la matríz identidad está implícita).